CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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Q.H. Ngô, N. Gargava, S. Goodenough, J.-Y. Enjalbert, P. Simonnet.

CIP seminar,
Friday conversations:
For this seminar, please have a look at Slide CCRT[n] & ff.

Outline

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

CCRT[22] MRS and the outer world III.1

PBW and rewriting techniques.

1. In the preceding weeks, we have considered the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (1)$$

2. This identity, formulated with a basis of Lie polynomials and its dual holds true, not only for other bases but also with other Lie algebras then the free one.

3. At first, one must pass from a basis of the Lie algebra in question $\mathfrak{g}$ (if it exists) to a basis of its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and this is the realm of the PBW (Poincaré-Birkhoff-Witt) theorem
(see [22], and [5], Ch I §2.7 Th 1).

4. In fact, WLOG, we can prove the mechanism of this rewriting technique in a much more general framework.

Back to the basics: $\mathcal{U}(\mathfrak{g})$.

First contact with PBW and how combinatorics on the words is convenient here.

5. A first version: that of MO.

6. **PBW Theorem** (source [22]). –
Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k, with an ordered basis $x_1 < x_2 < \dots < x_n$.
We define the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ to be the free noncommutative algebra $k\langle x_1, \dots, x_n\rangle$ modulo the relations $(x_i x_j - x_j x_i = [x_i, x_j])$.
The Poincaré–Birkhoff–Witt (PBW) theorem states that $\mathcal{U}(\mathfrak{g})$ has a basis consisting of lexicographically ordered monomials i.e. monomials of the form $x_1^{e_1} x_2^{e_2} \dots x_n^{e_n}$. Checking that this basis spans $\mathcal{U}(\mathfrak{g})$ is trivial, so the work lies in showing that these monomials are linearly independent.

7. **Here, we have to enlarge this framework** in two directions: firstly, we need infinite dimensional Lie algebras (for example the Free Lie algebra with its Lyndon basis) and secondly $\mathbf{k}$ must be a ring (e.g. $\mathbf{k} = \mathcal{H}(\Omega)$).

Back to the basics: $\mathcal{U}(\mathfrak{g})/2$

8. Let us make the terms crystal clear.
9. A **k**-Lie algebra $\mathfrak{g}$ is a **k**-algebra $(\mathfrak{g}, [\bullet, \bullet])$ (in general non associative and non-unital) which satisfies two identities (i.e. forall $x, y, z \in \mathfrak{g}$)
   a. $[x, x] = 0$ (Alternation)
   b. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity)
10. From [9.a], one infers that the bracket is antisymmetric i.e. identically $[y, x] = -[x, y]$. A system of generators being given, computations within a Lie algebra is determined by an alternate table (below the table of $sl_2(\mathbf{k})$)

First factor

$[\bullet, \bullet]$	e	h	f
e	0	$-2e$	h
h	$2e$	0	$-2f$
f	$-h$	$2f$	0

11. From [9.b and 10], one infers this (equivalent) form of [9.b]

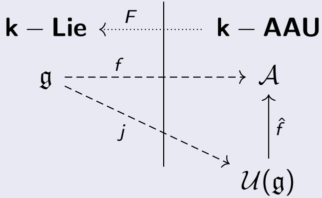
$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \tag{2}$$

which means that $ad_x : \ y \mapsto [x, y]$ is a derivation of $(\mathfrak{g}, [\bullet, \bullet])$.

Back to the basics: $\mathcal{U}(\mathfrak{g})/3$

12. In fact, the definition given in point 6 was just local (`field + finite dimension`) and, as was said in 7, we need the general definition. We give it in the CCRT spirit.

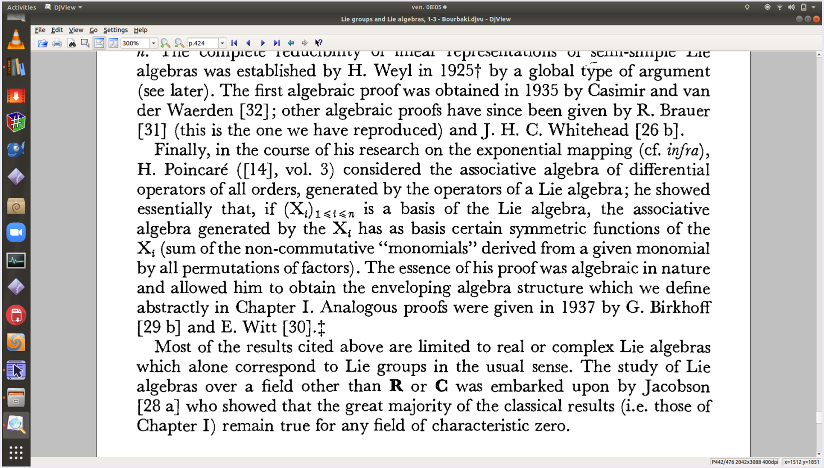
13. $\mathcal{U}(\mathfrak{g})$ is the solution of an universal problem (**k** is a ring).



14. Now, below
    1. History of PBW
    2. The general PBW Theorem.

Origins of PBW

15. Let us read



n. The complete reducibility of linear representations of semi-simple Lie algebras was established by H. Weyl in 1925† by a global type of argument (see later). The first algebraic proof was obtained in 1935 by Casimir and van der Waerden [32]; other algebraic proofs have since been given by R. Brauer [31] (this is the one we have reproduced) and J. H. C. Whitehead [26 b].

Finally, in the course of his research on the exponential mapping (cf. *infra*), H. Poincaré ([14], vol. 3) considered the associative algebra of differential operators of all orders, generated by the operators of a Lie algebra; he showed essentially that, if $(X_i)_{1 \leqslant i \leqslant n}$ is a basis of the Lie algebra, the associative algebra generated by the X_i has as basis certain symmetric functions of the X_i (sum of the non-commutative "monomials" derived from a given monomial by all permutations of factors). The essence of his proof was algebraic in nature and allowed him to obtain the enveloping algebra structure which we define abstractly in Chapter I. Analogous proofs were given in 1937 by G. Birkhoff [29 b] and E. Witt [30].‡

Most of the results cited above are limited to real or complex Lie algebras which alone correspond to Lie groups in the usual sense. The study of Lie algebras over a field other than **R** or **C** was embarked upon by Jacobson [28 a] who showed that the great majority of the classical results (i.e. those of Chapter I) remain true for any field of characteristic zero.

16. and see how to calculate on one of the simplest noncommutative infinite dimensional Lie algebra: that of (polynomial) vector fields on the line.

Example: computations in the Witt algebra.

17 The Witt algebra is

1. The Lie algebra of meromorphic vector fields on the Riemann sphere that are holomorphic except at two fixed points
2. The complexification of the Lie algebra of polynomial vector fields on a circle
3. The Lie algebra of derivations $\mathfrak{Der}(\mathbb{C}[z, z^{-1}])$.

18 a classical basis of it is $L_n = -z^{n+1}\frac{d}{dz}$

19 its law is $[L_m, L_n] = (m-n)L_{m+n}$

20 and a central extension of it is the **Virasoro Lie algebra** (see [21])

21 the general PBW below will teach us that the corresponding AAU of differential operators has a basis formed, with $n_1 < \ldots < n_k$, by the monomials

$$L_{n_1}^{\alpha_1} \cdots L_{n_k}^{\alpha_k} = (-1)^k \left(z^{n_1+1}\frac{d}{dz}\right)^{\alpha_1} \cdots \left(z^{n_k+1}\frac{d}{dz}\right)^{\alpha_k} \tag{3}$$

The general PBW theorem.

Theorem (Poincaré-Birkhoff-Witt [5, 22])

Let $\mathbf{k}$ *be a ring and* $\mathfrak{g}$ *a* $\mathbf{k}$*-Lie algebra which is free as a* $\mathbf{k}$*-module. Let* $B = (b_i)_{i \in I}$ *be a (totally) ordered basis of* $\mathfrak{g}$ *(as a free* $\mathbf{k}$*-module, then). We now construct a family within* $\mathcal{U}(\mathfrak{g})$*, using the following multiindex notation.*
For every $\alpha \in \mathbb{N}^{(I)}$*, we set*

$$B^\alpha = b_{i_1}^{\alpha(i_1)} \cdots b_{i_n}^{\alpha(i_n)} \in \mathcal{U} \tag{4}$$

where $\{i_1, \cdots, i_n\} \supset \mathrm{supp}(\alpha)$ *(and* $i_1 < \cdots < i_n$*). One sees easily that this product does not depend on the choice of* $\{i_1, \cdots, i_n\}$ *provided that* $\mathrm{supp}(\alpha)$ *be a subset of it. Then*

1. $\mathcal{U}(\mathfrak{g})$ *is free as a* $\mathbf{k}$*-module.*
2. $(B^\alpha)_{\alpha \in \mathbb{N}^{(I)}}$ *is a basis of* $\mathcal{U}(\mathfrak{g})$*.*

22 **Remarks.** – i) One can reverse the order to obtain a *decreasing form*.
ii) We can realize a model of the PBW basis within $\mathbf{k}\langle B \rangle$ using increasing words and this is precisely what we will now do.

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Q: Nice proofs of the Poincaré–Birkhoff–Witt theorem

The Poincaré–Birkhoff–Witt (PBW) **theorem** states that $U(\mathfrak{g})$ has a basis consisting **of** lexicographically ordered monomials i.e. monomials **of the** form $x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$. ... What other **proofs of** PBW are there out there? Are there **nice** reformulations **of the** above **proof** from a different perspective, such as one that emphasizes **the** universal property **of** $U(\mathfrak{g})$? ...

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asked Feb 3 '12 by user332

15 votes | 1 answer

Q: So, did Poincaré prove PBW or not?

It is known that **Poincaré**, at least, invented something that can be called **Poincaré-Birkhoff-Witt theorem** (PBW **theorem**) in 1900. ... Tran, Poincaré's **proof of the** so-called **Birkhoff-Witt theorem**, Rev. Histoire Math., 5 (1999), pp. 249-284, also arXiv:math/9908139. ...

lie-algebras | ho.history-overview | universal-algebra

asked Sep 8 '11 by darij grinberg

https://mathoverflow.net

Classical rewriting [5], Ch I §2.7 Lemma 1

M.)

Lemma 1. *For every integer $p \geqslant 0$, there exists a unique homomorphism f_p of the* K-*module* $\mathfrak{g} \otimes_{\mathrm{K}} \mathrm{P}_p$ *into the* K-*module* P *satisfying the following conditions:*

(A_p) $\quad f_p(x_\lambda \otimes z_{\mathrm{M}}) = z_\lambda z_{\mathrm{M}}$ *for* $\lambda \leqslant \mathrm{M}$, $z_{\mathrm{M}} \in \mathrm{P}_p$;

(B_p) $\quad f_p(x_\lambda \otimes z_{\mathrm{M}}) - z_\lambda z_{\mathrm{M}} \in \mathrm{P}_q$ *for* $z_{\mathrm{M}} \in \mathrm{P}_q$, $q \leqslant p$;

(C_p) $\quad f_p(x_\lambda \otimes f_p(x_\mu \otimes z_{\mathrm{N}})) = f_p(x_\mu \otimes f_p(x_\lambda \otimes z_{\mathrm{N}})) + f_p([x_\lambda, x_\mu] \otimes z_{\mathrm{N}})$

for $z_{\mathrm{N}} \in \mathrm{P}_{p-1}$. *(The terms appearing in* (C_p) *are meaningful by* (B_p)*.)*

Moreover, the restriction of f_p to $\mathfrak{g} \otimes \mathrm{P}_{p-1}$ *coincides with* f_{p-1}.

The last assertion follows from the others since the restriction of f_p to $\mathfrak{g} \otimes_{\mathrm{K}} \mathrm{P}_{p-1}$ satisfies conditions (A_{p-1}), (B_{p-1}) and (C_{p-1}). We shall prove the existence and uniqueness of f_p by induction on p. For $p = 0$, condition (A_0) gives $f_0(x_\lambda \otimes 1) = z_\lambda$ and conditions (B_0) and (C_0) are then obviously satisfied. Suppose now that the existence and uniqueness of f_{p-1} are proved. We show that f_{p-1} admits a unique extension f_p to $\mathfrak{g} \otimes_{\mathrm{K}} \mathrm{P}_p$ satisfying conditions (A_p), (B_p) and (C_p).

We must define $f_p(x_\lambda \otimes z_{\mathrm{M}})$ for an increasing sequence M of p elements.

If $\lambda \leqslant \mathrm{M}$, the value is given by condition (A_p). Otherwise, M can be written uniquely in the form (μ, N), where $\mu < \lambda$, $\mu \leqslant \mathrm{N}$. Then

$$z_{\mathrm{M}} = z_\mu z_{\mathrm{N}} = f_{p-1}(x_\mu \otimes z_{\mathrm{N}})$$

23 **Ex9. –**

Let $(\mathcal{X}, <)$ be an ordered alphabet[a].
(Statisitics "inv"). For each $w \in \mathcal{X}^*$, we set

$$inv(w) := \#\{(i,j) \in [1, \ldots, n]^2 \mid i < j \text{ and } w[i] > w[j]\} \tag{5}$$

1) Enumerate $inv(w)$ (for example in the style of [36]) for $\mathcal{X} = \{a, b\},\ a < b$ and the words of length ≤ 3 (15 words).
2) Let $\mathcal{X}^{\uparrow}$ be the set of increasing words (i.e. defined by $inv(w) = 0$). Give the set $\mathcal{X}^{\uparrow}_{\leq 3}$ (6 words) and $\mathcal{X}^{\uparrow}_{\leq 4}$.
3) a) For $\mathcal{X} = \{x_1, \ldots, x_k\}$ a finite alphabet indexed in increasing order and $x \in \mathcal{X}$, we set $nb(x) = j$ such that $x = x_j$. For $w \in \mathcal{X}^{\uparrow}_n$ show that the function

$$\phi_n :\ j \mapsto nb(w[j]) + j - 1 \tag{6}$$

is strictly increasing. Give it for a^3, a^2b, ab^2, b^3 $\begin{pmatrix} 0 & 1 & 2 \\ a & b & b \\ 1 & 2 & 2 \end{pmatrix}$

[a]Unless otherwise specified, an ordered alphabet will always be understood as totally ordered.

Intermezzo: increasing words.

24 **Ex9 cont'd.** –

3) b) What is its domain, codomain ?

4) Prove that $w \mapsto \{\phi_n(w)[i]\}_{i=1}^{n}$ is a one-to-one correspondence between $\mathcal{X}_n^{\uparrow}$ and

$$\binom{\{1,\ldots,k+n-1\}}{n}$$

where, for a set E and $k \in \mathbb{N}$, the binomial $\binom{E}{k}$ is the set of k-subsets of E.

5) For $|\mathcal{X}| = k$, what is the cardinality of $\mathcal{X}_n^{\uparrow}$?

25 **Ex10.** – (Straightening principle)

Let $\mathcal{X}_{UnderDiag} \subset \mathcal{X}^2$ be the set of pairs (x, y) such that $x > y$ and B an **arbitrary map** $B : \ \mathcal{X}_{UnderDiag} \to \mathbf{k}.\mathcal{X} = \mathbf{k}^{(\mathcal{X})}$ defined by its structure constants

$$B(x,y) = \sum_{z\in\mathcal{X}} \gamma_{x,y}^{z}\, z \tag{7}$$

Intermezzo: increasing words/2

26 **Ex10 cont'd. –**
We define $\mathbf{k}\langle\mathcal{X}^{\uparrow}\rangle$ as the set of polynomials with support in $\mathcal{X}^{\uparrow}$ and consider maps, linear on the right

$$\rho: \mathcal{X} \times \mathbf{k}\langle\mathcal{X}^{\uparrow}\rangle \to \mathbf{k}\langle\mathcal{X}^{\uparrow}\rangle$$

such that

1. $\rho(x, u) = xu$ if $xu \in \mathcal{X}^{\uparrow}$
2. $\rho(x, yv) = \rho(B(x, y), v) + \rho(y, \rho(x, v)))$ otherwise $(x > y)$

1) Prove that such a ρ exists and is unique (imitate the method of [5], Ch I §2.7 Lemma 1).
2) One defines an "elementary straightening" by

1. $es(w) = w$ if $w \in \mathcal{X}^{\uparrow}$
2. $es(pxys) = p(B(x, y) + yx)s$ if $x > y$ is the first inversion

Prove that the sequence es^n is locally stationary and recompute ρ with $\lim_{n\to+\infty}(es^n)$.
3) Reprove the PBW using the straightening principle of (**Ex10**).

27. **Ex10** is due to M.P. Schützenberger [34].
28. For example, using the sl_2 table

First factor

$[\bullet,\bullet]$	e	h	f
e	0	$-2e$	h
h	$2e$	0	$-2f$
f	$-h$	$2f$	0

$e < h < f$ and $B(x,y) = [x,y]$, we get the following execution (with $\phi(x,u) = x|u$)

$$f|eh = [f,e]|h + e|(f|h) =$$
$$-h|h + e|([f,h]|1) + e|(h|(f|1)) =$$
$$-h^2 + 2(e|f) + ehf = -h^2 + 2ef + ehf \quad (8)$$

A small tribute to MPS or Marco as we used to call him

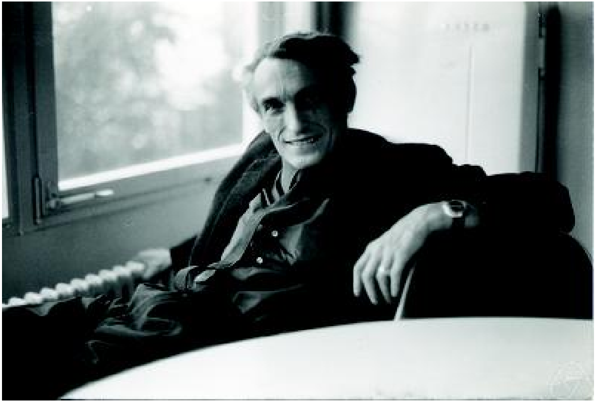


Figure: Marcel-Paul Schützenberger at Oberwolfach (1973)[1]

[1] Contrary to 1972 (Wikipedia)

General setting for MRS.

Theorem, GHED, DG, HNM [15]

Let **k** be a $\mathbb{Q}$-algebra and $\mathfrak{g}$ be a Lie algebra which is free as a **k**-module. Let us fix an ordered basis $B = (b_i)_{i\in I}$ (where the ground set $(I, <)$ is totally ordered) of $\mathfrak{g}$. To construct the associated PBW basis of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, we use the following multiindex notation. For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha(i_1)} \cdots b_{i_n}^{\alpha(i_n)} \in \mathcal{U} \tag{9}$$

where $\{i_1, \cdots, i_n\} \supset \mathrm{supp}(\alpha)$ (and $i_1 < \cdots < i_n$).
Consider the linear coordinate forms $B_\beta \in \mathcal{U}^\vee$ defined by

$$\langle B_\beta | B^\alpha \rangle = \delta_{\alpha,\beta}. \tag{10}$$

We will also use the elementary multiindices $e_i \in \mathbb{N}^{(I)}$ defined for all $i \in I$ by $e_i(j) = \delta_{i,j}$.

General setting for MRS/2

Theorem cont'd

Then:[a]

1. We have

$$B_\alpha \circledast B_\beta = \frac{(\alpha+\beta)\,!}{\alpha\,!\,\beta\,!} B_{\alpha+\beta} \tag{11}$$

   and

$$B_{\alpha(i_1)e_{i_1}+\cdots+\alpha(i_k)e_{i_k}} = \frac{B_{e_{i_1}}^{\circledast\alpha(i_1)} \circledast \cdots \circledast B_{e_{i_k}}^{\circledast\alpha(i_k)}}{\alpha(i_1)\,!\cdots\alpha(i_k)\,!}. \tag{12}$$

2. The following infinite product identity holds:

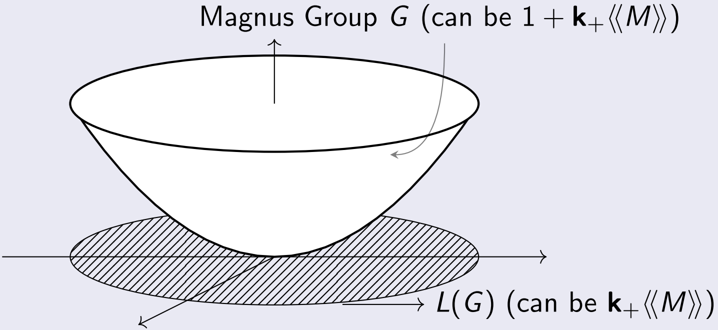
$$Id_{\mathcal{U}} = \circledast_{i\in I}^{\rightarrow} \; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} = \prod_{i\in I}^{\rightarrow} \; e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} \tag{13}$$

   within $End(\mathcal{U})$.

[a] We use the notation $\alpha!$ for $\alpha \in \mathbb{N}^{(I)}$; this is the product $\alpha! = \prod_{i\in I} \alpha_i!$.

MRS Identity within different Magnus and Hausdorff (i.e. groups of characters) groups.

29. Here, we want to serve the generality in order to taylor a more general tool.



Concluding remarks

1. In this CCRT[22] MRS variation 1, we have focused on PBW and combinatorics on words.
2. We returned back to the basics and the categorical definition of $\mathcal{U}(\mathfrak{g})$.
3. We saw the general PBW (Recent works and classical rewriting).
4. We showed that this rewriting (initial on words) can be better formulated on increasing words.
5. This gives us a implementable version of the straightening algorithm.
6. To end with, the general setting for MRS was provided.

Concluding remarks/2

7. Remains to exploit this setting on different Lie groups as the Magnus group for Shuffle and Stuffle.
8. The mechanics of this computations is the object of a forthcoming paper "Kleene stars in shuffle algebras" (see [15]).
9. These two groups and their systems of coordinate (through MRS) being set, we need a "bridge formula" to match them.
10. Remains to
    1. Prove that this formula is separating (i.e. transversality is sufficient)
    2. To what extent identification of local coordinates reflects the relations between their coefficients.

THANK YOU FOR YOUR ATTENTION !

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